Midterm

① Prove Z^n is analytic:

$Z = re^{i\theta} \Rightarrow Z^n = r^n e^{in\theta}$

$Z^n = r^n \cos n\theta + i\, r^n \sin n\theta$

$u = r^n \cos n\theta \qquad v = r^n \sin n\theta$

$u_r = n r^{n-1} \cos n\theta \qquad v_r = n r^{n-1} \sin n\theta$

$u_\theta = -n r^n \sin n\theta \qquad v_\theta = n r^n \cos n\theta$

$u_r = \frac{1}{r} v_\theta \longrightarrow (1)$

$v_r = -\frac{1}{r} u_\theta \longrightarrow (2)$

∴ Cauchy Reiman conditions are right

∴ Z^n is analytic ∀ (entire function)

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* $\frac{d}{dz}[\ln z] = \frac{1}{z}$

$$\lim_{\Delta z \to 0} \frac{\ln(z+\Delta z) - \ln z}{\Delta z} = \lim_{\Delta z \to 0} \frac{1}{\Delta z} \ln\left(\frac{z+\Delta z}{z}\right)$$

$$\lim_{\Delta z \to 0} \ln\left(\frac{z+\Delta z}{z}\right)^{\frac{1}{\Delta z}} = \lim_{\Delta z \to 0} \ln\left(1+\frac{\Delta z}{z}\right)^{\frac{z}{\Delta z} \cdot \frac{1}{z}}$$

$$\frac{1}{z} \lim_{\Delta z \to 0} \ln\left(1+\frac{1}{z/\Delta z}\right)^{\frac{z}{\Delta z}} = \frac{1}{z} \ln e = \frac{1}{z}$$

∵ $\left(1+\frac{1}{n}\right)^n = e$

OR prove by paths

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$\ln z = \ln r + i\theta$

$u = \ln r \qquad v = \theta$

$u_r = \frac{1}{r} \qquad v_\theta = 1$

$u_\theta = 0 \qquad v_r = 0$

$u_r = \frac{1}{r} v_\theta \quad , \quad v_r = -\frac{1}{r} u_\theta$

Cauchy Reiman condition are correct

2 $\ln z$ is analytic

2 $f'(z) = (u_r + i v_r) e^{-i\theta} = \frac{e^{-i\theta}}{r}$

Note $z = r e^{i\theta}$ at $\theta = 0 \quad z = r$

OR $r = 1 \quad z = e^{i\theta}$

$\therefore f'(z) = \frac{1}{z}$

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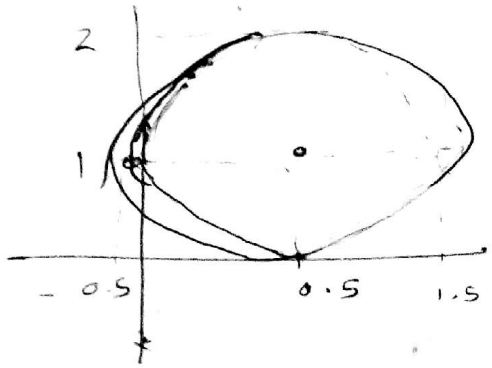
[2] $\oint \frac{z+1}{z^2+1} dz$



$|z - (0.5 + i)| = 1$

$z^2 + 1 = 0$

$z = \pm i \rightarrow$ singular points

$z = +i$

$2\pi i f(z_0) = \oint \frac{f(z)}{z - z_0} dz$

$f(z) = \frac{z+1}{z+i}\Big|_{z=i} = \frac{i+1}{2i} = \frac{1}{2}(1 - i)$

$\therefore \oint = 2\pi i \frac{1}{2}(1 - i) = \pi i + \pi$

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(3) $u = \sqrt{r}\cos\left(\frac{\theta}{2}\right)$

$u_r = \frac{1}{2\sqrt{r}}\cos\left(\frac{\theta}{2}\right)$

$u_\theta = -\frac{\sqrt{r}}{2}\sin\left(\frac{\theta}{2}\right)$

$u_r = \frac{1}{r}v_\theta$, $v_r = -\frac{1}{r}u_\theta$

$v = \int r u_r \, d\theta = \int \frac{r}{2\sqrt{r}}\cos\left(\frac{\theta}{2}\right) d\theta$

$v = \frac{2\sqrt{r}}{2}\sin\left(\frac{\theta}{2}\right) + f(r)$

$v_r = \frac{1}{2\sqrt{r}}\sin\left(\frac{\theta}{2}\right) + f'(r) = 0 + \frac{1}{r}\frac{\sqrt{r}}{2}\sin\left(\frac{\theta}{2}\right)$

$\therefore f'(r) = 0$

$\therefore f(r) = C$

$\therefore v = \sqrt{r}\sin\left(\frac{\theta}{2}\right) + C$

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[4] $\csc\left(\frac{\pi}{z}\right)$ $\quad |z| < \frac{1}{\pi}$

$\therefore \csc(z)$ $\quad 0 < |z| < \pi$

$\therefore \csc\left(\frac{1}{z}\right)$ $\quad 2\pi > |z| > \pi$

Laurent $= \csc\left(\frac{\pi}{z}\right)$

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$\csc(z)$ $\quad 0 < z < \pi$

$$\csc(z) = \frac{1}{\sin(z)}$$

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} - \cdots$$

$$\frac{1}{\sin(z)} = \frac{1}{z - \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!}}$$

قسمة ←

$$\frac{1}{z} + \frac{z}{3!} + z^3\left(\frac{1}{(3!)^2} - \frac{1}{5!}\right)$$

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \;\Big)\; 1$$

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!}$$

$$\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!}$$

$$\frac{z^2}{3!} - \frac{z^4}{(3!)^2} + \frac{z^6}{5!(3!)}$$

$$z^4\left(\frac{1}{(3!)^2} - \frac{1}{5!}\right)$$

$$\therefore \csc(z) = \frac{1}{z} + \frac{z}{3!} + z^3\left(\frac{1}{(3!)^2} - \frac{1}{5!}\right) + \cdots$$

$0 < |z| < \pi$

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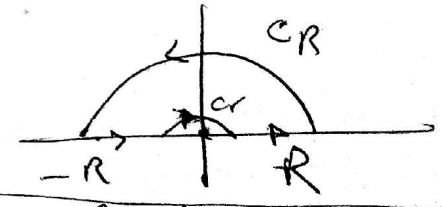
⑤ $\displaystyle\int_0^\infty \frac{\cos 3x}{x^2}\,dx$

$\frac{\text{even}}{\text{even}} \longrightarrow$ even $\therefore \displaystyle\int_0^\infty \cdots = \frac{1}{2}\int_{-\infty}^{\infty}$

$$\oint \frac{e^{i3z}}{z^2}\,dz = 0$$



$$\int_{C_R} = \int_0^\pi \left|\frac{e^{i3z}}{z^2}\right| |dz| \le \frac{e^{-3R\sin t}}{R^2}\,\pi R \xrightarrow{R\to\infty} 0$$

$z = Re^{it}$ $\quad dz = iRe^{it}\,dt$

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$$\int_{-\infty}^{\infty} \frac{e^{i3x}}{x^2}\,dx$$

$$\int_{C_1} = \frac{1}{2}\oint \frac{e^{i3z}}{z^2}\,dz = -\frac{1}{2}\,2\pi i \left.\frac{d}{dz}\left[e^{i3z}\right]\right|_{z=0}$$

(عكس عقارب الساعة)

$$= -\pi i \times [i3] = +3\pi$$

$$\therefore 0 = \int_{-\infty}^{\infty} \frac{e^{i3x}}{x^2}\,dx + 3\pi$$

$$-3\pi = \int_{-\infty}^{\infty} \frac{\cos 3x}{x^2} + i\int_{-\infty}^{\infty} \frac{\sin 3x}{x^2}\,dx$$

$$\therefore \int_{0}^{\infty} \frac{\cos 3x}{x^2} = -\frac{3\pi}{2}$$

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